Intermediate inflation in the Jordan-Brans-Dicke theory
Work in collaboration with Sergio del Campo [JCAP 01(2011)013]

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August 4, 2011
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The observed universe

Standard Model

Inflationary Model
Standard cosmological model (SCM)

- General Relativity:
  \[ G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \]

  \( \kappa^2 = 8\pi G, \ c = \hbar = k_B = 1 \) and \( T_{\mu\nu} \) : energy-momentum tensor

- Homogeneity + isotropy \( \Rightarrow \) Friedmann-Robertson-Walker metric:
  \[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

  \( a(t) \) : the scale factor, \( k \) : the curvature for spatial sections

- The field equations for the SCM are:
  \[ H^2(t) + \frac{k}{a^2(t)} = \frac{\kappa^2}{3} \rho(t) \quad \text{and} \quad \dot{\rho}(t) + 3H(t) [\rho(t) + p(t)] = 0 \]

  \( H \equiv \frac{\dot{a}}{a} \) : cosmic expansion rate, \( \rho \) : energy density and \( p \) : pressure
Inflationary cosmological model


- Accelerated expansion
  \[ \ddot{a} > 0 \Rightarrow p < -\frac{\rho}{3} \]

- Scalar field \( \phi \)
  \[
  \rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V(\phi) \\
  p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V(\phi)
  \]

- Slow-roll regime
  \[ \varepsilon \equiv -\frac{\dot{H}}{H^2} < 1 \quad \text{and} \quad \eta \equiv -\frac{\dddot{H}}{\dot{H}H} < 1 \]

- Field equations \((k = 0)\)
  \[
  H^2 = \frac{k^2}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \\
  \dddot{\phi} + 3H\ddot{\phi} + V'(\phi) = 0
  \]

- It is convenient to define the number of e-folds as:
  \[ N = \ln \left( \frac{a(t)}{a_i} \right) \]
Linear perturbations in cosmological models

- Our universe is not completely homogeneous and isotropic,

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \bar{g}_{\mu\nu} :\text{FRW metric and } h_{\mu\nu} \ll \bar{g}_{\mu\nu} \]

\[ \delta G^\mu_\nu = \kappa^2 \delta T^\mu_\nu \]

- We use the longitudinal gauge in a flat FRW background:

\[ ds^2 = -(1 + 2\Phi)dt^2 + a(t)^2 \left[ (1 - 2\Phi)\delta_{ij} + h_{ij} \right] dx^i dx^j \]

- The comoving curvature perturbation \( \mathcal{R} \):

\[ \mathcal{R} \equiv \Phi + \frac{2\rho}{3(\rho + p)} \left( \frac{\dot{\Phi}}{H} + \Phi \right) \quad \text{and} \quad \dot{\mathcal{R}} = -\frac{H}{\dot{\rho}} \left( -\frac{k^2\Phi}{a^2} + 3\dot{H}S \right) \]

\[ S \equiv H \left( \frac{\delta p}{\rho} - \frac{\delta \rho}{\dot{\rho}} \right) \text{ is called the entropy perturbation} \]

- At large scales (\( k \ll aH \)) and for one scalar field \( \dot{\mathcal{R}} \rightarrow 0 \)
Intermediate Inflation

- Intermediate inflation is an exact solution to the inflationary field equations in the Einstein theory [J.D Barrow, Phys.Lett.B 235 40 (1990)]
- Intermediate inflation considers a scale factor of the form:
  \[ a(t) = a_0 e^{At^f} \text{ where } 0 < f < 1 \text{ and } A > 0 \]
- In the slow-roll regime the potential takes the form:
  \[ V(\phi) \propto \phi^n, \text{ where } n \equiv -4 \left( \frac{1 - f}{f} \right) < 0 \]
- We note that this potential does not have a minimum!
- In the Einstein theory we get a scale invariant spectrum for a particular value of \( n \) for intermediate inflation
The Jordan-Brans-Dicke theory (JBD)

- The Jordan-Brans-Dicke theory considers the variation of the universal gravitational constant $G$, represented in this context by a scalar field
  

- In 1989 it was studied an inflationary realization of this theory: extended inflation. This model failed, however it motivates inflationary realizations with more than one scalar field
  

- The Brans-Dicke parameter of this theory is constrained by observational tests for consistency of General Relativity
  
Two conformally equivalent actions

There are two conformally equivalent actions from which we can describe the model: [A.A. Starobinsky & J. Yokoyama, arXiv:9502002 [gr-qc]]

\[
S = \int \sqrt{-g} d^4x \left[ \frac{1}{2\kappa^2} R + \frac{1}{2} g^\mu\nu \partial_\mu \chi \partial_\nu \chi + \frac{1}{2} e^{-\gamma \kappa \chi} g^\mu\nu \partial_\mu \phi \partial_\nu \phi - e^{-\beta \kappa \chi} V(\phi) \right],
\]

\[
S = \int \sqrt{-\hat{g}} d^4x \left[ \frac{\Phi_{BD}}{16\pi} \hat{R} + \frac{\omega^2}{16\pi \Phi_{BD}} \hat{g}^\mu\nu \partial_\mu \Phi_{BD} \partial_\nu \Phi_{BD} + \frac{1}{2} \hat{g}^\mu\nu \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]
\]

where $\chi$ and $\phi$ are the dilaton and inflaton fields respectively, $\omega$ is the Brans-Dicke parameter, $\Phi_{BD}$ is the Brans-Dicke field, $\gamma = \frac{1}{\sqrt{\omega + \frac{3}{2}}} = \frac{\beta}{2}$

Both actions are conformally related by $g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}$ and
$\Omega^2 \equiv \frac{\kappa^2}{8\pi} \Phi_{BD} \equiv e^{\gamma \kappa \chi}$.

In the limit $\omega \to \infty$ we recover the Einstein theory.

The observational constraints suggest that $\omega > 3500 \leftrightarrow \beta < 0.03$
Field equations in the Einstein frame

\[
\ddot{\chi} + 3H \dot{\chi} + \frac{\gamma \kappa}{2} e^{-\gamma \kappa \chi} \dot{\phi}^2 - \beta \kappa e^{-\beta \kappa \chi} V(\phi) = 0 \\
\ddot{\phi} + 3H \dot{\phi} - \gamma \kappa \dot{\chi} \dot{\phi} + e^{(\gamma - \beta) \kappa \chi} V'(\phi) = 0 \\
3H^2 - \kappa^2 \left( \frac{1}{2} \dot{\chi}^2 + \frac{1}{2} e^{-\gamma \kappa \chi} \dot{\phi}^2 + e^{-\beta \kappa \chi} V(\phi) \right) = 0
\]

Interaction

\[
\rho_\phi = \frac{1}{2} e^{-\gamma \kappa \chi} \dot{\phi}^2 + e^{-\beta \kappa \chi} V(\phi) \\
p_\phi = \frac{1}{2} e^{-\gamma \kappa \chi} \dot{\phi}^2 - e^{-\beta \kappa \chi} V(\phi) \\
\rho_\chi = p_\chi = \frac{1}{2} \dot{\chi}^2
\]

\[
3H^2 = \kappa^2 (\rho_\phi + \rho_\chi) \\
\dot{\rho}_\phi + 3H (\rho_\phi + p_\phi) = Q \\
\dot{\rho}_\chi + 3H (\rho_\chi + p_\chi) = -Q
\]
The field equations in the slow-roll regime take the following form:

\[
3H \dot{\chi} - \beta \kappa e^{-\beta \kappa \chi} V(\phi) = 0,
\]
\[
3H \dot{\phi} + e^{(\gamma - \beta) \kappa \chi} V'(\phi) = 0,
\]
\[
3H^2 - \kappa^2 e^{-\beta \kappa \chi} V(\phi) = 0.
\]

By considering the ansatz \( a(t) = a_i \left( 1 + \frac{A}{p} t^f \right)^p \) we find:

\[
V(\phi) = V_0 \phi^n, \quad \phi(t) = \left( \frac{8 \sqrt{V_0} (1 - f) t}{\sqrt{3 \kappa f^2}} \right)^{\frac{1}{2}}, \quad \chi(a) = \beta \kappa \ln \left( \frac{a}{a_i} \right) + \chi_i,
\]

where \( p \equiv \frac{2}{\beta^2}, \ 0 < f \equiv \frac{4}{4-n} < 1 \) and \( A > 0 \).
The figures shows 250 e-folds of inflation for $f = 0.7$ and $\beta = 0.01$. For the last 100 e-folds of inflation the error in using the slow-roll approximation is less than 1%.
Field equations in the slow-roll regime

- It is required at least 50 e-folds of inflation to solve the problems of the standard model and to push the perturbations to observable scales. The exact number depends on the particular model.

- An intermediate inflationary period needs an additional mechanism to end the inflationary era [J.D. Barrow, A.R. Liddle & C. Pahud, Phys.Rev.D 74 127305 (2006)]. We consider that this mechanism starts when have passed $N_T$ e-folds since the beginning of inflation. We normalize $\chi$ in such a way that after $N_T$ e-folds the value of the field $\chi$ becomes zero, consequently: $\chi_i = -\frac{\beta}{\kappa} N_T$.

- We will assume that inflation starts when $\epsilon = 1$, where:

\[
\begin{align*}
\epsilon &= -\frac{\dot{H}}{H^2} = 1 - \frac{\ddot{a}}{aH^2} = \frac{\beta^2}{2} + e^{\gamma \kappa \chi} \frac{1}{2\kappa^2} \left( \frac{V'}{V} \right)^2, \\
t_i &= \left( \frac{2(1 - f)}{A(2f - \beta^2)} \right)^{\frac{1}{7}}, \quad V_0 = \frac{3A_i^2 f^4 \kappa^2 e^{\beta \kappa \chi b}}{4^{3 - \frac{3}{7}} (1 - f)^2} \left( \frac{(1 - f)(2f - \beta^2)}{f^2 (2 - \beta^2) \kappa^2} \right)^{\frac{2}{7}}.
\end{align*}
\]
Scalar perturbations in the slow-roll regime

The perturbed field equations come from introducing the metric in the longitudinal gauge to the Einstein equations

\[ \dot{\Phi} + H\Phi = \frac{k^2}{2} \left( \dot{\chi}\delta\chi + e^{-\gamma\kappa\chi}\dot{\phi}\delta\phi \right), \]

\[ \ddot{\delta\chi} + 3H\dot{\delta\chi} + \left( \frac{k^2}{a^2} - \frac{(\gamma\kappa)^2}{2} e^{-\gamma\kappa\chi}\dot{\phi}^2 + (\beta\kappa)^2 e^{-\beta\kappa\chi}V(\phi) \right) \delta\chi + \gamma\kappa e^{-\gamma\kappa\chi}\dot{\phi}\dot{\delta\phi} - \beta\kappa e^{-\beta\kappa\chi}V'(\phi)\delta\phi = 2(\ddot{\chi} + 3H\dot{\chi})\Phi + \dot{\Phi}\dot{\chi} + 3\dot{\Phi}\dot{\chi} + \gamma\kappa e^{-\gamma\kappa\chi}\dot{\phi}^2\Phi, \]

\[ \ddot{\delta\phi} + (3H - \gamma\kappa\dot{\chi})\dot{\delta\phi} + \left( \frac{k^2}{a^2} + e^{(\gamma - \beta)\kappa\chi}V''(\phi) \right) \delta\phi - \gamma\kappa\dot{\phi}\dot{\delta\chi}\]

\[ + (\gamma - \beta)\kappa V'(\phi)e^{(\gamma - \beta)\kappa\chi}\delta\chi = 2(\ddot{\phi} + 3H\dot{\phi})\Phi + \dot{\Phi}\dot{\phi} + 3\dot{\Phi}\dot{\phi} - 2\gamma\kappa\dot{\phi}\dot{\phi}\Phi. \]

\( k \) stands for the Fourier space decomposition.

In order to find a solution, we consider the slow-roll regime:

\( \dot{\Phi} \ll H\Phi, \ddot{\delta\chi} \ll H\dot{\delta\chi}, \ddot{\delta\phi} \ll H\dot{\delta\phi}, \dot{\delta\chi} \ll H\delta\chi, \delta\phi \ll H\delta\phi \)
Besides of the slow-roll regime for perturbations we consider the large scale approximation ($k \ll aH$):

$$
\Phi = \frac{\kappa^2}{2H} \left( \dot{\chi} \delta \chi + e^{-\gamma \kappa \chi} \dot{\phi} \delta \phi \right) = \frac{\beta \kappa}{2} \delta \chi - \frac{V'(\phi)}{2V(\phi)} \delta \phi,
$$

$$
3H\dot{\chi} + (\beta \kappa)^2 e^{-\beta \kappa \chi} V(\phi) \delta \chi - \beta \kappa e^{-\beta \kappa \chi} V'(\phi) \delta \phi = 2\beta \kappa e^{-\beta \kappa \chi} V(\phi) \Phi,
$$

$$
3H\dot{\phi} + e^{(\gamma - \beta) \kappa \chi} V''(\phi) \delta \phi + (\gamma - \beta) \kappa V'(\phi) e^{(\gamma - \beta) \kappa \chi} \delta \chi = -2e^{(\gamma - \beta) \kappa \chi} V'(\phi) \Phi.
$$

The solutions to these equations are given by:

$$
\frac{\delta \chi}{\dot{\chi}} = \frac{C_1}{H} \frac{1}{H} - \frac{C_3}{H}, \quad \frac{\delta \phi}{\dot{\phi}} = \frac{C_1}{H} \frac{1}{H} + \frac{C_3}{H} (e^{-\gamma \kappa \chi} - 1),
$$

$$
\Phi = -C_1 \frac{\dot{H}}{H^2} + C_3 \left( \frac{1 - e^{\gamma \kappa \chi}}{2\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 - \frac{\beta^2}{2} \right)
$$

where $C_1$ and $C_3$ are integration constants.
The comoving curvature perturbation remains:

\[ \mathcal{R} = C_1 - C_3 W(\phi, \chi), \]

where the function \( W(\phi, \chi) \) is defined as:

\[ W(\phi, \chi) \equiv 1 - \left( e^{\gamma \kappa \chi} + \beta^2 \kappa^2 \left( \frac{V(\phi)}{V'(\phi)} \right)^2 \right)^{-1}. \]
The comoving curvature perturbation for the numerical solution and the slow-roll solution. The error in using the slow-roll approximation is less than 5%
Assuming that $\mathcal{R}$ is maintained approximately constant for large scales:

$$P_{\mathcal{R}}(k) = \frac{4\pi k^3}{(2\pi)^3} \langle |C_1|^2 \rangle = \left[ \frac{H^2 e^{2\gamma\kappa\chi}}{(2\pi)^2} \left( (e^{-\gamma\kappa\chi} - 1)^2 \frac{H^2}{\chi^2} + \frac{H^2 e^{\gamma\kappa\chi}}{\dot{\phi}^2} \right) \right]_{k=aH},$$

where:

$$C_1 = H e^{\gamma\kappa\chi} \left( \frac{\delta \chi}{\dot{\chi}} \left( e^{-\gamma\kappa\chi} - 1 \right) + \frac{\delta \phi}{\dot{\phi}} \right).$$

and the expectation values for the perturbations of the fields $\delta \phi$ y $\delta \chi$ are given by:

$$\langle \delta \phi^* (k) \delta \phi (k') \rangle = \frac{H^2}{2k^3} e^{\gamma\kappa\chi} \delta^3 (k - k') \quad \text{and} \quad \langle \delta \chi^* (k) \delta \chi (k') \rangle = \frac{H^2}{2k^3} \delta^3 (k - k')$$

which have to be evaluated for $k = aH$. 

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By considering that the amplitude of the power spectrum in a given scale \( k_\ast = 0.002 \text{[Mpc]}^{-1} \) assumes the value
\[
P_R(k_\ast) = 2.43 \times 10^{-9} \quad \text{[N. Jarosik et al., arXiv:1001.4744 [astro-ph]]},
\]
we can get the values of the parameters in the model \( A, V_0, t_i \).

The stripes represent, from darkest to lightest, \( f = 0.4; 0.6; 0.8 \). The dashed line corresponds to a scale \( k = 0.002 \text{Mpc}^{-1} \) leaving the horizon after 10 e-folds since the beginning of inflation for different set of parameters.
In addition to the scalar curvature perturbation, tensor perturbations can also be generated from quantum fluctuations during inflation. The tensor perturbations do not couple to matter and consequently they are only determined by the dynamics of the background metric, so the standard results for the evolution of tensor perturbations of the metric remains valid. The two independent polarizations evolve like minimally coupled massless fields with spectrum:

$$P_T(k) = 8\kappa^2 \left(\frac{H}{2\pi}\right)^2_{k=aH}$$

The tensor to scalar ratio is given by:

$$r \equiv \frac{P_T}{P_R}$$
Spectral index and tensor perturbations

\[ n_s(k_*) \equiv 1 + \frac{d \ln P_R}{d \ln k} = 1 - 4\epsilon_* + \eta_* + \frac{\beta^2}{2} Z_{1*}, \]

\[ r(k_*) \equiv \frac{P_T}{P_R} = \frac{8(1 - f)Z_0^2\beta^2}{(Z_* - Z_0)^2 + f(2Z_*(-1 + Z_0) - Z_0^2) + Z_*\beta^2}, \]

where

\[ Z \equiv e^{\frac{N \beta^2}{2}} \left( 2 - \beta^2 \right), \quad Z_0 \equiv e^{\frac{N_T \beta^2}{2}} \left( 2 - \beta^2 \right), \]

\[ Z_1 \equiv 1 - \frac{fZ}{f(Z - 2) + \beta^2} + \frac{Z}{Z - 2f + \beta^2} + \frac{Z^2 + (f - 1)Z_0^2}{(Z - Z_0)^2 + f(2Z(Z_0 - 1) - Z_0^2) + Z\beta^2}, \]
Results

Trajectories for different values of the parameters $\beta$ and $f$ in the $n_s - r$ plane. We compare with the WMAP data (five and seven years). The two contours correspond to 68%CL and 95%CL. The left panel shows $N_T = 250$ e-folds whereas the right panel is for $N_T = 60$ e-folds.
Final remarks

- The model of intermediate inflation in the JBD theory seems to be consistent with the observational data for some parameters.
- The solution is restricted in the total number of efolds allowed.
- We have guaranteed (by numerical integration) the existence of the slow-roll regime for the set of parameters of interest.
- This model inflation does not finish by itself, it is necessary some additional mechanism which could modify the predictions here exposed.
- The mechanism of reheating could modify the predictions here exposed.