PALATINI APPROACH to MODIFIED $f(R)$ GRAVITY and its BI-METRIC STRUCTURE

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The action that defines an $f(R)$ gravity is given by

$$ S = \int d^4x \sqrt{-g} \left[ \frac{f(R)}{2\kappa^2} + \mathcal{L}_m \right], \quad (1) $$

where $\kappa^2 = 8\pi G$, $g$ is the determinant of the metric tensor, $\mathcal{L}_m$ is the Lagrangian density for the matter fields and

$$ R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \left( \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\alpha\sigma} \Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\nu\sigma} \Gamma^\sigma_{\mu\alpha} \right). \quad (2) $$

Einstein's General Relativity

In Einstein's General Relativity $f(R) = R$ and the connections are given \textit{a priori} as the Christoffel symbols of a metric $g$:

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However, metric determines distances and connection the parallel transport of vectors; in principle these are independent fields [T. Levi-Civita: \textit{The Absolute Differential Calculus}].
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Palatini Approach to Modified $f(R)$ Gravity

The Equations of Motion

In the Palatini variational approach the metric and the affine connections are treated as independents and the variation is taken with respect to both, giving us:

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} = 8\pi G T_{\mu\nu} ,$$  \hspace{1cm} (4)

$$\nabla_\alpha (f' \sqrt{-g} g^{\mu\nu}) = 0 , \quad \text{where} \quad f' = df/dR.$$  \hspace{1cm} (5)

- Eqs. (4) $\Rightarrow$ Modified Einstein’s Equations.
- Eqs. (5) $\Rightarrow$ Give us the connections:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} h^{\alpha\sigma} (\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) ,$$  \hspace{1cm} (6)

where $h_{\mu\nu} = f' g_{\mu\nu}$ is a new conformal metric. Let us remark that:

- The dynamics of $\Gamma$ identifies a new metric $h_{\mu\nu}$ in the manifold.
- It is the connection $\Gamma$ which determines the tensor curvature of spacetime.
- If $f(R) = R$ the field equations (5) $\Rightarrow$ $\Gamma^\alpha_{\mu\nu} = \{^{\alpha}_{\mu\nu}\}_g$, so
- the Levi-Civita connection of $g$ (Eq. (3)) is no longer an assumption a priori, it is the outcome of field equations!
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where \( h_{\mu\nu} = f' g_{\mu\nu} \) is a new conformal metric. Let us remark that:

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- the Levi-Civita connection of \( g \) (Eq. (3)) is no longer an assumption \textit{a priori}, it is the outcome of field equations!
In the Palatini approach we are faced with the following questions:

- What is the role of the metric $h$ on the spacetime manifold?
- Since the gravitational field is represented by the connections, the free-fall particles follow $g$-geodesics or $h$-geodesics?
- How many $h_{\mu\nu} = f' g_{\mu\nu}$ ($f' \neq 0$) exist for a given $g_{\mu\nu}$?
- In a given manifold $(M, g)$, is there some evidence of the "apparent"metric $h$?

For a beautiful discussion about the first two issues, see M. Capozziello et al.: Found. Phys. 39, 1161 (2009). The third question has been discussed in a number of papers:


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For an excellent review about Palatini approach do \( f(R) \) theories see G.J. Olmo, Int. J. Mod. Phys. D 20, 413 (2011); arXiv:1101.3864 gr-qc.

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Conformal Transformations

Let \((M, g)\) be a spacetime with metric \(g\), \((X, \phi_t)\) a smooth vector field on \(M\) with the associated diffeomorphisms \(\phi_t\). \(X\) is a **Conformal Vector Field** (also called **Conformal Killing Vector**) if the diffeomorphisms \(\phi_t\) preserve the metric up to a conformal factor: \(\phi_t^* g = \Omega(x) g\), for some positive function \(\Omega\) [G.S. Hall: *Symmetries and Curvature Structure in General Relativity*, World Scientific (2004)].

**Lie Derivatives**

This can be cast in terms of the Lie Derivative of the metric tensor as

\[
\mathcal{L}_X g_{\mu\nu} = 2\phi_A g_{\mu\nu}
\]  

(7)

where \(\phi_A\) is called the conformal function of \(X_A\) and

\[
\mathcal{L}_X g_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu X^\alpha + g_{\mu\alpha} \partial_\nu X^\alpha
\]

(8)

We are interested in the **Conformal Vector Fields** \(X_A\), and respective conformal functions, possibly admitted by the flat \((k = 0)\) FLRW spacetime such that

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where \(f' = df/dR \neq 0\) are the derivatives of \(f(R)\) theories of gravity.
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Conformal Transformations & Lie Derivatives

Conformal Vector Fields in \((k = 0)\) FLRW

We write the metric of the flat FLRW spacetime as

\[
d s^2 = a^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right)
\]

(10)

where \(\eta\) is the conformal time; \(d\eta = dt/a(t)\). For this metric there are 9 Conformal Vectors [Y. Choquet-Bruhat et al.: Analysis, Manifolds and Physics; Norh-Holland (1977)]:

\[
\begin{align*}
X_0 &= \partial_\eta \\
K_0 &= -2\eta X_4 - (x_\alpha x^\alpha)X_0 \\
X_i &= x_i \partial_\eta + \eta \partial_i \\
X_4 &= x^\alpha \partial_\alpha \\
K_i &= 2x_i X_4
\end{align*}
\]

(11)

where \(x^\alpha = (\eta, x, y, z)\) and \(x_\alpha = (-\eta, x, y, z)\).
From Eqs. (9) we determine the "conformal function" $f'(R)$ associated with each Conformal Vector Field $X_A$ ($A = 1 \ldots 9$):

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\begin{array}{|c|c|}
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\text{Conformal Vector} & f' = \frac{df}{dR} \text{ (associated } f(R) \text{ theory)} \\
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X_0 & d \ln a / d\eta \\
X_i & x_i d \ln a / d\eta \\
X_4 & 1 + \eta d \ln a / d\eta \\
K_0 & -2\eta - (\eta^2 + x^2 + y^2 + z^2) d \ln a / d\eta \\
K_i & 2x_i (1 + \eta d \ln a / d\eta) \\
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CONCLUSIONS

- In the Palatini approach to $f(R)$ gravity, beside the metric $g$, another metric $h$ is involved, which gives the connection $\Gamma$. These two metrics are related by a conformal transformation such that $h_{\mu\nu} = f'g_{\mu\nu}$.
- The new metric $h$ is directly linked to conformal Killing symmetries preexistent in the manifold $(M, g)$.
- We relate the conformal Killing vectors in FLRW flat spacetime to $f(R)$ theories in the Palatini formalism.
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