Explicit expressions for optical scalars in gravitational lensing from general matter sources

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I CosmoSul, Rio De Janeiro; Agosto 1, 2011
Content

1. Introduction
   - Standard gravitational lensing notation

2. Some observations on the standard gravitational lensing expressions

3. Integrated expansion and shear
   - General equations: The geodesic deviation equation
   - Approximation method for solving the geodesic deviation equation
   - Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation
   - The thin lens approximation
   - Spherically symmetric lenses
   - Spherically symmetric lenses

4. Two simple examples

5. Summary
1 Introduction
   - Standard gravitational lensing notation

2 Some observations on the standard gravitational lensing expressions

3 Integrated expansion and shear
   - General equations: The geodesic deviation equation
   - Approximation method for solving the geodesic deviation equation
   - Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation
   - The thin lens approximation
   - Spherically symmetric lenses
   - Spherically symmetric lenses

4 Two simple examples

5 Summary
Gravitational lensing in astrophysics

- Gravitational lensing has become a significant tool to make progress in our knowledge on the matter content of our Universe. In particular, there is a large number of works that use gravitational lensing techniques in order to know how much mass are in galaxies or clusters of galaxies.

- One of the most exciting results was to reaffirm the need for some kind of dark matter, that appears to interact with the barionic matter only through gravitation.

- The question in which there is yet not general agreement is on the nature of this dark matter. The most common conception is that it is based on collisionless particles [Wei08], and where the pressures are negligible. However in the context of cosmological studies, one often recurs to models of dark matter in terms of scalar fields or spinors fields.
Standard gravitational lensing notation

In many astrophysical situations, the gravitational effects on light rays is weak, and the source and observer are far away from the lens, therefore they are studied under the formalism of weak field and thin gravitational lenses.

In the framework of weak field gravitational lensing the lens equation reads

\[
\beta^a = \theta^a - \frac{d_{ls}}{d_s} \alpha^a.
\]  

(1)
The differential of this equation can be written as

$$\delta \beta^a = A^a_b \delta \theta^b,$$

where the matrix $A^a_b$ is in turn expressed by

$$A^a_b = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix},$$

where the optical scalars $\kappa$, $\gamma_1$ and $\gamma_2$, are known as convergence $\kappa$ and shear components $\{\gamma_1, \gamma_2\}$, or $\gamma = \gamma_1 + i\gamma_2$ and have the information of distortion of the image of the source due to the lens effects.
1 Introduction
   - Standard gravitational lensing notation

2 Some observations on the standard gravitational lensing expressions

3 Integrated expansion and shear
   - General equations: The geodesic deviation equation
   - Approximation method for solving the geodesic deviation equation
   - Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation
   - The thin lens approximation
   - Spherically symmetric lenses
   - Spherically symmetric lenses

4 Two simple examples

5 Summary
Introduction: Standard gravitational lensing notation

It is somehow striking that in most astronomical works on gravitational lensing, it is assumed that the lens scalars and deflection angle, can be obtained from a Newtonian-like potential function. These expressions although are easy to use, have some limitations:

- They neglect more general distribution of energy-momentum tensor $T_{ab}$, in particular they only take into account the timelike component of this tensor. In this way they severely restrict the possible candidates to dark matter that can be studied with these expressions.
- They are not expressed in terms of gauge invariant quantities.
- Since these expressions are written in terms of a potential function, it is not easily seen how different components of $T_{ab}$ contribute in the generation of these images.
- Most of them assume from the beginning that thin lens is a good approximation.

△ We extend the work appearing in standard references on gravitational lensing and present new expressions that do not suffer from the limitations mentioned above.
△ We present gauge invariant expressions for the optical scalars and deflection angle for some general class of matter distributions.
△ In this work we study gravitational lensing over a flat background.
Content

1 Introduction
   - Standard gravitational lensing notation

2 Some observations on the standard gravitational lensing expressions

3 Integrated expansion and shear
   - General equations: The geodesic deviation equation
   - Approximation method for solving the geodesic deviation equation
   - Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation
   - The thin lens approximation
   - Spherically symmetric lenses

4 Two simple examples

5 Summary
General equations: The geodesic deviation equation

Let us consider the general case of a null geodesic starting from the position $p_s$ (source) and ending at $p_o$ (observer). Let us characterize the tangent vector as $\ell = \frac{\partial}{\partial \lambda}$; so that

$$\ell^b \nabla_b \ell^a = 0; \quad (4)$$

that is, $\lambda$ is an affine parameter.

We can now consider also a continuous set of nearby null geodesics. This congruence of null geodesics can be constructed in the following way. Let $S$ be a two dimensional spacelike surface (the source image) such that the null vector $\ell$ is orthogonal to $S$. Next we can generalize $\ell$ to be a vector field in the vicinity of the initial geodesic in the following way: let the function $u$ be defined so that it is constant along the congruence of null geodesics emanating orthogonally to $S$ and reaching the observing point $p_o$. Then, without loss of generality we can assume that

$$\ell_a = \nabla_a u; \quad (5)$$

which implies that the congruence has zero twist.
We can complete to a set of null tetrad, so that $m^a$ and $\bar{m}^a$ are tangent to $S$. At other points $m^a$ is chosen so that it is tangent to the surfaces $u =$constant and $\lambda =$constant. Then a deviation vector at the source image can be expressed by

$$\varsigma^a = \varsigma \bar{m}^a + \bar{\varsigma} m^a.$$  \hspace{1cm} (6)

In order to propagate this deviation vector along the null congruence one requires, that its Lie derivate vanishes along the congruence; that is

$$\mathfrak{L}_\ell \varsigma^a = 0.$$  \hspace{1cm} (7)

Using the GHP notation one can show that the geodesic deviation equation can be written as

$$0 = \mathfrak{D}(\varsigma) + \varsigma \rho + \bar{\varsigma} \sigma;$$

where $\mathfrak{D}$ is the well behaved derivation of type $\{1,1\}$ in the direction of $\ell$ (the null geodesic vector of the congruence).

Defining $\chi$ by

$$\chi = \begin{pmatrix} \varsigma \\ \bar{\varsigma} \end{pmatrix};$$  \hspace{1cm} (8)
the equation for $\varsigma$ can be written as

$$\ell(\ell(\mathcal{X})) = -Q\mathcal{X};$$  \hspace{0.5cm} (9)

where $Q$ is given by

$$Q = \left( \begin{array}{cc} \Phi_{00} & \Psi_0 \\ \bar{\Psi}_0 & \Phi_{00} \end{array} \right);$$  \hspace{0.5cm} (10)

with

$$\Phi_{00} = -\frac{1}{2}R_{ab}\ell^a\ell^b,$$  \hspace{0.5cm} (11)

and

$$\Psi_0 = C_{abcd}\ell^a\ell^b\ell^c\ell^d.$$  \hspace{0.5cm} (12)

Therefore, this form of the equation only involves curvature quantities.
Approximation method for solving the geodesic deviation equation

Let us first transform to a first order differential equation. Defining $\nu$ to be

$$\nu \equiv \frac{d\chi}{d\lambda}; \quad (13)$$

and

$$x \equiv \begin{pmatrix} \chi \\ \nu \end{pmatrix}; \quad (14)$$

one obtains

$$\ell(x) = \frac{dx}{d\lambda} = \begin{pmatrix} \nu \\ -Q\chi \end{pmatrix} = A x; \quad (15)$$

with

$$A \equiv \begin{pmatrix} 0 & I \\ -Q & 0 \end{pmatrix}. \quad (16)$$

Equation (15) can be re-expressed in integral form, which gives

$$x(\lambda) = x_0 + \int_{\lambda_0}^{\lambda} A(\lambda') x(\lambda') d\lambda'. \quad (17)$$
Approximation method for solving the geodesic deviation equation II

The complete linear iteration is

\[
X_3(\lambda) = \left( \begin{array}{c}
II - \int_{\lambda_0}^{\lambda} \int_{\lambda_0}^{\lambda'} Q'' \, d\lambda'' \, d\lambda' \\
- \int_{\lambda_0}^{\lambda} Q' \, d\lambda' \\
II - \int_{\lambda_0}^{\lambda} (\lambda'' - \lambda_0) Q'' \, d\lambda'' \, d\lambda'
\end{array} \right) X_0. \tag{18}
\]

Now in order to integrate the geodesic deviation equation, we must choose the correct initial conditions. In the case of light rays belonging to the past null cone of the observer and intersecting \( S \) at the source, this initial conditions are \( \mathcal{X} = 0 \) and \( \mathcal{V} \neq 0 \); since one can think the beam, starts backwards in time from the observer position, and so initially has vanishing departure, but with nonzero expansion and shear. Therefore in the linear approximation one has

\[
\mathcal{X}(\lambda) = \left( (\lambda - \lambda_0)II - \int_{\lambda_0}^{\lambda} (\lambda - \lambda')(\lambda' - \lambda_0) Q' \, d\lambda' \right) \mathcal{V}(\lambda_0); \tag{19}
\]

and

\[
\mathcal{V}(\lambda) = \left( II - \int_{\lambda_0}^{\lambda} (\lambda' - \lambda_0) Q' \, d\lambda' \right) \mathcal{V}(\lambda_0). \tag{20}
\]
Approximation method for solving the geodesic deviation equation III

If the metric were flat \((Q = 0)\), in order to get a deviation vector constructed from \(\mathcal{X}_1\), defined as \(\mathcal{X}'\) evaluated at \(\lambda_s = \lambda_0 + d_s\), one must choose as initial condition

\[\mathcal{N}(\lambda_0) = \frac{1}{(\lambda_s - \lambda_0)} \mathcal{X}'(\lambda_s = \lambda_0 + d_s) = \frac{1}{d_s} \mathcal{X}_1.\]  

(21)

Using a complex displacement \(\varsigma\) of unit modulus; namely \(\varsigma = e^{i\varphi}\), to represent the deviation vector, one can express the equation in the form

\[\varsigma_s(\varphi) = \left[1 - \frac{1}{d_s} \int_0^{d_s} \mathcal{X}'(d_s - \lambda') \Phi_{00}(\lambda') d\lambda' - \left(\frac{1}{d_s} \int_0^{d_s} \mathcal{X}'(d_s - \lambda') \Psi_0(\lambda') d\lambda'\right) e^{-2i\varphi}\right] e^{i\varphi}.\]  

(22)
Optical scalar in terms of the curvature

In order to compare with the standard representation of the lens scalar we note that the original deviation vector in the source will be given by the previous equation i.e.

\[
\left( \begin{array}{c}
\varsigma_s \\
\bar{\varsigma}_s
\end{array} \right) = \left( I - \int_0^{ds} \frac{\chi'(d_s - \chi')}{ds} Q' d\chi' \right) \left( \begin{array}{c}
\varsigma_o \\
\bar{\varsigma}_o
\end{array} \right) ; \tag{23}
\]

if we make the following decomposition into real and imaginary part,

\[
\varsigma_o = \varsigma_{oR} + i\varsigma_{oI} , \quad (24)
\]
\[
\varsigma_s = \varsigma_{sR} + i\varsigma_{sI} , \quad (25)
\]
\[
\Psi_0 = \Psi_{0R} + i\Psi_{0I} ; \quad (26)
\]

we obtain from eq.(23) that

\[
\varsigma_{sR} = \left( 1 - \int_0^{ds} \frac{\chi'(d_s - \chi')}{ds} \left( \Phi_{00}' + \Psi_{0R}' \right) d\chi' \right) \varsigma_{oR} - \left( \int_0^{ds} \frac{\chi'(d_s - \chi')}{ds} \Psi_{0I}' d\chi' \right) \varsigma_{oI} ,
\]
\[
\varsigma_{sI} = \left( 1 - \int_0^{ds} \frac{\chi'(d_s - \chi')}{ds} \left( \Phi_{00}' - \Psi_{0R}' \right) d\chi' \right) \varsigma_{oI} - \int_0^{ds} \frac{\chi'(d_s - \chi')}{ds} \Psi_{0I}' d\chi' \varsigma_{oR} . \tag{27}
\]
Note also that in principle the integration must be made through the actual geodesic followed by a photon in its path from the source to observer. However the last expressions are valid only in the limit where the linear approximation is valid. If one considers a linear perturbation from flat spacetime, then the curvature components $\Phi_{00}$ and $\Psi_0$ would be already of linear order. Then, in the context of weak field gravitational lensing, it is consistent to consider a null geodesic in flat spacetime; since the actual null geodesic can be thought as a null geodesic in flat spacetime plus some corrections of higher orders.

Now, in order to compare with the usual expressions for the lens scalars $\kappa, \gamma_1$ and $\gamma_2$, let us recall that they are defined via the relation eq.(2); but since it is a linear relation, one can relate the deviation vectors by the same matrix, namely

$$\varsigma^i_s = A^i_j \varsigma^j_o; \quad (28)$$

where $\{\varsigma^i_s, \varsigma^i_o\}$ are the spatial vector associated with $\{\varsigma_s, \varsigma_o\}$ respectively. Therefore, by replacing into eq.(28), we obtain

$$\varsigma_sR = (1 - \kappa - \gamma_1)\varsigma_oR - \gamma_2 \varsigma_oI; \quad (29)$$

$$\varsigma_sI = -\gamma_2 \varsigma_oR + (1 - \kappa + \gamma_1)\varsigma_oI; \quad (30)$$
Optical scalar in terms of the curvature III

By taking real and imaginary part of previous equation one obtains

\[ \kappa = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Phi'_{00} \, d\lambda', \quad (31) \]

\[ \gamma_1 = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Psi'_{0R} \, d\lambda', \quad (32) \]

\[ \gamma_2 = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Psi'_{0I} \, d\lambda'. \quad (33) \]

- These expressions for the weak field lens quantities are explicitly gauge invariant, since they are given in terms of the curvature components,

- Note that these expressions are valid for any weak field gravitational lens on a perturbed flat spacetime, without restriction on the size of the lens compared with the other distances.

- Note that the equations for the shears can be written as

\[ \gamma_1 + i \gamma_2 = \frac{1}{d_s} \int_0^{d_s} \lambda' (d_s - \lambda') \Psi'_0 \, d\lambda'. \quad (34) \]
The thin lens approximation I

The general case

If we assume a thin lens, then $\Phi_{00}$ and $\Psi_0$ will be sharply peaked around $\lambda = d_l$ where is located the lens. Then, the expressions for the lens scalars are reduced to

$$\kappa = \frac{d_l d_{ls}}{d_s} \hat{\Phi}_{00},$$

(35)

$$\gamma_1 + i \gamma_2 = \frac{d_l d_{ls}}{d_s} \hat{\Psi}_0,$$

(36)

where

$$\hat{\Phi}_{00} = \int_0^{d_s} \Phi_{00} d\lambda,$$

(37)

$$\hat{\Psi}_0 = \int_0^{d_s} \Psi_0 d\lambda,$$

are the projected curvature scalars along the line of sight.
Let us define

$$\hat{\Psi}_0(J) = -e^{2i\vartheta} \hat{\psi}_0(J),$$

(38)

and

$$\gamma_1 + i\gamma_2 = -\gamma e^{2i\vartheta};$$

(39)

then one has

$$\kappa = \frac{d_l s d_l}{d_s} \hat{\Phi}_{00}(J),$$

(40)

$$\gamma = \frac{d_l s d_l}{d_s} \hat{\psi}_0(J).$$

(41)
Deflection angle in terms of projected Ricci and Weyl Scalars

We wish now to express the deflection angle in terms of the curvature scalars. If we define the components of $\alpha = (\alpha^1, \alpha^2)$ as

$$\langle \alpha^i \rangle = \alpha(J) \left( \frac{z_0}{J}, \frac{x_0}{J} \right); \quad (42)$$

Then, it can be shown then, that the optical scalars can be written in terms of $\alpha(J)$ as

$$\kappa = \frac{1}{2} \frac{d l_s d l}{d s} \left( \frac{d \alpha}{d J} + \frac{\alpha(J)}{J} \right), \quad (43)$$

$$\gamma_1 = \frac{1}{2} \frac{d l_s d l}{d s} \cos(2 \vartheta) \left( \frac{d \alpha}{d J} - \frac{\alpha(J)}{J} \right), \quad (44)$$

$$\gamma_2 = \frac{1}{2} \frac{d l_s d l}{d s} \sin(2 \vartheta) \left( \frac{d \alpha}{d J} - \frac{\alpha(J)}{J} \right). \quad (45)$$

It is interesting to note that

$$\kappa - \gamma_1 \cos(2 \vartheta) - \gamma_2 \sin(2 \vartheta) = \frac{d l d l_s}{d s} \frac{\alpha(J)}{J}; \quad (46)$$
Finally from the last expression it is deduced that the deflection angle for axially symmetric spacetimes is given by

$$\alpha(J) = J(\hat{\Phi}_{00}(J) + \hat{\psi}_0(J))$$

(47)

This constitutes a very simple equation for the bending angle expressed in terms of the gauge invariant curvature components in compact form.
Expressions for the bending angle in terms of energy-momentum components and $M(r)$

A stationary spherically symmetric spacetime can be expressed in terms of the standard line element

$$ds^2 = a(r)dt^2 - b(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2);$$  (48)

where it is convenient to define $\Phi(r)$ and $m(r)$ from

$$a(r) = e^{2\Phi(r)},$$  (49)

and

$$b(r) = \frac{1}{1 - \frac{2M(r)}{r}}.$$  (50)
Spherically symmetric lenses I

For our purpose, it is more convenient to use a null coordinate system to describe the spherically symmetric geometry. Let us introduce then, a function

\[ u = t - r^*; \]  

(51)

where \( r^* \) is chosen so that \( u \) is null. Then by inspection of equation (48) one can see that

\[ du = dt - \frac{dr^*}{dr} dr = dt - \sqrt{\frac{b}{a}} dr; \]  

(52)

since then one has

\[ ds^2 = a du^2 + 2\sqrt{ab} du dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(53)

It is natural to define the principal null direction \( \tilde{\ell}_P \) from

\[ \tilde{\ell}_P = du; \]  

(54)

which implies that the vector is

\[ \tilde{\ell}_P^a = g^{ab} du_b = \frac{1}{\sqrt{ab}} \left( \frac{\partial}{\partial r} \right)^a. \]  

(55)
We complete to a null tetrad with

\[ \tilde{n}_P = \frac{\partial}{\partial u} + U A \frac{\partial}{\partial r}, \]  

(56)

with the complex null vector

\[ \tilde{m}_P = \frac{\sqrt{2} P_0}{r} \frac{\partial}{\partial \zeta}; \]  

(57)

in terms of the stereographic coordinate \( \zeta \). In these expressions

\[ A = \frac{1}{\sqrt{ab}}, \]  

(58)

and

\[ U = \frac{1}{2b A^2} = \frac{a}{2}. \]  

(59)

The energy momentum tensor can be given by

\[ T_{tt} = \varrho e^{2\Phi(r)}; \]  

(60)

\[ T_{rr} = \frac{P_r}{\left(1 - \frac{2M(r)}{r}\right)^2}; \]  

(61)
Spherically symmetric lenses III

\[ T_{\theta\theta} = P_t r^2; \quad (62) \]
\[ T_{\varphi\varphi} = P_t r^2 \sin(\theta)^2; \quad (63) \]

where we have introduced the notion of radial component \( P_r \) and tangential component \( P_t \), due to our general anisotropic assumption.

The \((t, t)\) component of the field equations implies

\[ \frac{dM}{dr} = 4\pi r^2 \varrho. \quad (64) \]

One can show that the Ricci scalars are given by

\[ \tilde{\Phi}_{00} = \frac{4\pi}{a} (\varrho + P_r), \quad (65) \]
\[ \tilde{\Phi}_{11} = \pi (\varrho - P_r + 2P_t), \quad (66) \]
\[ \tilde{\Phi}_{22} = a\pi (\varrho + P_r), \quad (67) \]

and the Weyl scalar

\[ \tilde{\Psi}_2 = \frac{4\pi}{3} (\varrho - P_r + P_t) - \frac{M}{r^3}. \quad (68) \]
When one made a tetrad transformation between the spherically symmetric tetrad and the tetrad adapted to the photon geodesic, one find that the function $\alpha(J)$ expressed in terms of the spherically symmetric null tetrad reads,

$$
\alpha(J) = J \int_{-d_I}^{d_L} \left[-\frac{3J^2}{r^2} \tilde{\Psi}_2 + \frac{2J^2}{r^2} (\tilde{\Phi}_{11} - \frac{1}{4} \tilde{\Phi}_{00}) + \tilde{\Phi}_{00}\right] dy.
$$

(69)

Note that in this case, the integration is on the coordinate $y$, instead of using arbitrary affine parameter. Also note that $r = \sqrt{J^2 + y^2}$.

This constitutes an important explicit relation for the bending angle in terms of the curvature as seen in an spherically symmetric frame; which is the natural frame for the sources of the gravitational lens.

In terms of the physical energy-momentum tensor it is obtained

$$
\alpha(J) = J \int_{-d_I}^{d_L} \left[\frac{3J^2}{r^2} \left(\frac{M(r)}{r^3} - \frac{4\pi}{3} \varrho(r)\right) + 4\pi (\varrho(r) + P_r(r))\right] dy ;
$$

(70)
Expressions for the lens scalars in terms of energy-momentum components and $M(r)$

$$
\kappa = \frac{4\pi d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \left[ \rho + P_r + \frac{J^2}{r^2} (P_t - P_r) \right] dy .
$$

$$
\gamma = \frac{d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \frac{J^2}{r^2} \left[ \frac{3M}{r^3} - 4\pi (\rho + P_t - P_r) \right] dy .
$$

These new expressions let us see explicitly the contributions of different components of the energy-momentum tensor on the optical scalars. One can see that a couple of terms disappear in the isotropic case in which $P_r = P_t$. 
1 Introduction
   - Standard gravitational lensing notation

2 Some observations on the standard gravitational lensing expressions

3 Integrated expansion and shear
   - General equations: The geodesic deviation equation
   - Approximation method for solving the geodesic deviation equation
   - Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation
   - The thin lens approximation
   - Spherically symmetric lenses
   - Spherically symmetric lenses

4 Two simple examples

5 Summary
Two simple examples

In order to show the application of our treatment of gravitational lens, we will consider next two standard models that are often used in representing the source of gravitational lenses.

A monopole mass (Schwarzschild)

As a simple example let there be a monopole distribution characterized by a mass $M$, therefore a simple computation gives $\tilde{\Phi}_{00} = 0$, and $\tilde{\Psi}_2 = -\frac{M}{r^3}$, then by considering that the observer and the source are far away, one can replace in the extremes of the integration (as is usually made) $d_s \to \infty$ and $d_l \to \infty$, then

$$\hat{\Phi}_{00} = 0, \quad (73)$$

$$\hat{\psi}_0 = -2 \int_0^\infty \frac{3J^2}{r^2} \tilde{\Psi}_2 dy = \frac{4M}{J^2}, \quad (74)$$

and by replacing into eqs.(40), (41) and (47), we readily obtain the well known results

$$\alpha(J) = \frac{4M}{J}, \quad (75)$$

$$\kappa = 0, \quad (76)$$

$$\gamma = \frac{d_l d_s}{d_s} \frac{4M}{J^2}. \quad (77)$$
The isothermal profile

One simple model of dark matter that is used to explain the rotation curves of galaxies is the isothermal profile, which is defined by the density function

$$\rho = \frac{v_c^2}{4\pi r^2}, \quad (78)$$

where $v_c$ is the circular velocity.

Since $v_c \ll c$, the pressures in this model are negligible. Then we obtain,

$$\hat{\Phi}_0 = \int_{-\infty}^{\infty} \frac{v_c^2}{r^2} dy = \frac{v_c^2 \pi}{J}, \quad (79)$$

$$\hat{\psi}_0 = \int_{-\infty}^{\infty} \frac{2J^2 v_c^2}{r^4} dy = \frac{v_c^2 \pi}{J}. \quad (80)$$
Two simple examples III

From these relations follow the well know results,

\[ \alpha = 2\pi v_c^2, \quad (81) \]
\[ \kappa = \frac{d_id_{ls}}{d_s} \frac{v_c^2 \pi}{J}, \quad (82) \]
\[ \gamma = \frac{d_id_{ls}}{d_s} \frac{\pi v_c^2}{J}. \quad (83) \]
1 Introduction
   - Standard gravitational lensing notation

2 Some observations on the standard gravitational lensing expressions

3 Integrated expansion and shear
   - General equations: The geodesic deviation equation
   - Approximation method for solving the geodesic deviation equation
   - Expressions for the lens optical scalar in terms of Weyl and Ricci curvature from geodesic deviation equation
   - The thin lens approximation
   - Spherically symmetric lenses

4 Two simple examples

5 Summary
We have presented explicit expressions for the bending angle and optical scalars in terms of all the components of the energy-momentum tensor for a variety of cases.

The incidence of the spacelike components of the energy-momentum tensor is not trivial.

This work can be extended of the study of sources with different structure and in a cosmological background.

It should be very interesting to study different models of dark matters in this context.

It might be that the description of dark matter needs for the consideration of the spacelike components of the energy-momentum tensor.